Math 275D Lecture 27 Notes

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December 12, 2019

1 The Martingale Central Limit Theorem

1.1 Motivation

The central limit theorem says something like

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1).$$

We have also learned about Donsker's theorem, which extends this idea to Brownian motion.

S_n is not necessarily small. If X_i are iid, then $\mathbb{E}[S_n] = n\,\mathbb{E}[X_i]$. The central limit theorem tells us that the fluctuation of S_n is much less than $n\,\mathbb{E}[X]$. We know that $\mathbb{E}[S_n] \sim n$ and $\operatorname{Var}(S_n) \sim n$. This is because

$$\mathbb{E}\left[\left(\sum_{i=1}^n X_i - \mathbb{E}[X_i]\right)^2\right] = \sum_{i,j} \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j]])],$$

where these terms are 0 if $i \neq j$ by independence. So the property comes from the increments $X_k = S_{k+1} - S_k$.

But the situation is more complicated for martingales. Let $S_1, S_2, \dots$ be a martingale. Then if

$$\frac{1}{n}\sum_{k=1}^n \mathbb{E}[(S_{k+1} - S_k)^2 \mid \mathcal{F}_k] \to c,$$

then

$$\frac{S_n}{\sqrt{n}} \sim \mathcal{N}(0,1).$$

In the iid case, $c = 1$. If we define $X_k = S_{k+1} - S_k$, then X_k and X_ℓ are not independent. So this result is nontrivial (and maybe even unintuitive). The idea is that under certain conditions, the X_k are independent.

1.2 The Markov chain CLT and martingale CLT

Theorem 1.1 (Martingale CLT)**.** *Let $\{S_k\}_k$ be a martingale, an dlet $X_k = S_{k+1} - S_k$. Suppose that*

$$\frac{\sum_{k=1}^{\lceil n \cdot t \rceil} \mathbb{E}[X_k^2 \mid \mathcal{F}_k]}{tn} \to 1 \qquad \forall t.$$

Then

$$\frac{S_{(n \cdot)}}{\sqrt{n}} \to B(\cdot).$$

Compare this to the Markov chain central limit theorem. Let $X_1, X_2, \dots$ be a Markov chain with a stationary distribution π. If we start the chain at π, then $X_1, X_2, \dots \overset{d}{=} X_2, X_3, \dots$; i.e. the sequence is **stationary**. The ergodic theorem says that $\frac{1}{n}\sum_{k=1}^n f(X_k) \to \mathbb{E}_\pi[f(X_1)]$; i.e. the sequence is **ergodic**.

Theorem 1.2 (Markov chain CLT)**.** *Let $\{X_n\}_n$ be an ergodic, stationary sequence with $\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] = 0$ and $\mathbb{E}[X_i^2] = 1$. Then*

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1).$$

The central limit theorem for Markov chains is a special case of the theorem for martingales. Let's prove this assuming the martingale CLT.

Proof. We want to show that

$$\frac{1}{n}\sum_k \mathbb{E}[X_k^2 \mid \mathcal{F}_{k-1}] \to 1.$$

Define $u_k := \mathbb{E}[X_k^2 \mid \mathcal{F}_{k-1}]$; this is also an ergodic, stationary sequence. So the property in the martingale CLT is satisfied. □

Now let's prove the martingale CLT.

Proof. If $\{S_n\}_n$ is a martingale with $S_0 = 0$ (and $\mathbb{E}[S_n^2] < \infty$), then we can define stopping times $T_1, \dots, T_n$ such that $(S_1, \dots, S_n) \overset{d}{=} (B_{T_1}, \dots, B_{T_n})$. This is a repeated use of Skorokhod's representation theorem. We then find that

$$\mathbb{E}[X_k^2 \mid \mathcal{F}_{k-1}] = \mathbb{E}[T_k - T_{k-1} \mid \mathcal{F}^B_{T_{k-1}}].$$

If $T_n \approx n$, we are done. We have $T_n = \sum_k T_k - T_{k-1}$ and $\sum_k \mathbb{E}[T_k - T_{k-1} \mid \mathcal{F}_{T_{k-1}}] = n$. If we can show that both are close, we will be done.

Let $\tau_k = T_k - T_{k-1}$, and let $V_k = \mathbb{E}[T_k - T_{k-1} \mid \mathcal{F}_{T_{k-1}}]$. We want to show that $\mathbb{E}[(\sum_k \tau_k - V_k)^2] = O(n)$. If $k < \ell$,

$$\mathbb{E}[(\tau_k - \mathbb{E}[\tau_k \mid \mathcal{F}_{T_{k-1}})(\tau_\ell - \mathbb{E}[\tau_\ell \mid \mathcal{F}_{T_{\ell-1}}])] = \mathbb{E}[\tau_k \tau_\ell] - \mathbb{E}[\tau_\ell\, \mathbb{E}[\tau_k \mid \mathcal{F}_{T_{k-1}}]] - \mathbb{E}[\tau_l\, \mathbb{E}[\tau_\ell \mid \mathcal{F}_{T_{\ell-1}}]$$

$$+ \mathbb{E}[\mathbb{E}[\tau_\ell \mid \mathcal{F}_{T_{\ell-1}}] \, \mathbb{E}[\tau_k \mid \mathcal{F}_{T_{k-1}}]]$$

The third term becomes $-\mathbb{E}[\tau_k \tau_\ell]$. We can calculate the other terms similarly.

$$= 0.$$

□